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Generalized Energy Representations for Current Groups

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The class of energy representations of the group $\mathscr{D}(X, G)$ of compactly supported smooth mappings from a manifold X into a compact semisimple Lie group G is substantially enlarged in the following two ways. One consists of when X has a Riemannian structure, to extend the work of Vershik, Gelfand, and Graev (*Compositio Math.* **35** (1977), 299–334; **42** (1981), 217–243), see also (Albeverio, Høegh-Krohn, and Testard, *J. Funct. Anal.* **41** (1981), 378–396); it is shown that each pair (ρ, M), where ρ is a strictly positive continuous density on X, and M a subbundle of the tangent bundle of X, gives rise to an energy representation $U_{\rho,M}$ of $\mathscr{D}(X, G)$ which is irreducible if $\dim(X) \geqslant 3$. The other, entirely new, does not require a Riemannian structure on X: a volume measure m on a smooth manifold X with Euler number $e(X) = 0$ being given, each pair (ρ, ξ), where ρ is a strictly positive continuous density on X and ξ a nonvanishing continuous vectorfield on X, gives rise to a new energy representation $\Pi^{\xi}_{\rho\, dm}$ of $\mathscr{D}(X, G)$ which is irreductible if $\dim(X) \geqslant 3$. Conditions of unitary equivalency of the $U_{\rho,M}$ as well as the $\Pi^{\xi}_{\rho\, dm}$ are given.

INTRODUCTION

Let $\mathscr{D}(X, G)$ be the set of compactly supported C^∞-mappings from a C^∞-manifold X into a compact semisimple Lie group G; endowed with a Schwartz topology and with the pointwise product it is a topological group called a current group.

(a) Until now, the only nonlocal and irreducible unitary representations of $\mathscr{D}(X, G)$ we knew were the ones constructed by Ismagilov [7] and Vershik *et al.* [10, 11] and which, ulteriorly, were the subject of extensive works by Albeverio *et al.* [1–3]. More precisely, when X is endowed with a Riemannian structure, to each pair (ρ, M) consisting of a C^∞ density ρ of X and of a subbundle M of the tangent bundle TX of X, the authors above

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associate a nonlocal unitary representation $U_{\rho,M}$ of order 1 of $\mathscr{D}(X,G)$. In [10, 11] the irreducibility of $U_{\rho,M}$, and conditions about unitary equivalence of such representations, are given when $\dim(X)\geqslant 4$, as well as the reducibility of $U_{\rho,M}$ when $\dim(X)=1$. The irreducibility of $U_{\rho,M}$ was extended to the case $\dim(X)\geqslant 3$ (and even, under some additional conditions, to the case $\dim(X)\geqslant 2$) in [2]. In all those works, some results about disjointness of Gaussian measures on Schwartz spaces take a very important place.

(b) Section I is devoted to the study of the disjointness of Gaussian measures (and their convolution with a Poisson measure) on the Schwartz space $\mathscr{D}'(\Omega)$, where Ω is a bounded cube in $\mathbb{R}^n$, $n\geqslant 3$, induced by Dirichlet forms [4, 6], the intervening coefficients being only continuous functions; it is a strengthening of the results of [2, Theorem 4.1 and corollaries] about Dirichlet forms with C^∞-coefficients.

(c) Section II is devoted to the enlargement of the class of the energy representations $U_{\rho,M}$, when X is endowed with a Riemannian structure, by taking densities ρ which are assumed only continuous. The results of Section I and works [2, 11] are then used to prove the irreducibility of $U_{\rho,M}$ when $\dim(X)\geqslant 3$ and to prove that U_{ρ_1,M_1} and U_{ρ_2,M_2} are unitarily equivalent if and only if $\rho_1=\rho_2$ and $M_1=M_2$; we call the $U_{\rho,M}$ generalized energy representations.

(d) When X is only assumed to be endowed with a volume measure m, the construction of the $U_{\rho,M}$ is not possible. Section III is devoted to the construction of a new kind of energy representations of $\mathscr{D}(X,G)$. More precisely it is shown that, when the Euler number $e(X)$ is zero, each pair (ρ,ξ), consisting of a strictly positive and locally $dm(x)$-measurable function ρ on X and of a continuous nonvanishing vectorfield ξ on X, gives rise to a nonlocal unitary representation $\Pi^{\xi}_{\rho\,dm}$ of order 1 of $\mathscr{D}(X,G)$ which appears as a partial energy representation. In particular it is shown that when X is endowed with a Riemannian structure inducing the volume measure m, then $\Pi^{\xi}_{\rho\,dm}$ is unitarily equivalent to the energy representation $U_{\rho_\xi,[\xi]}$, where $[\xi]$ is the subbundle of TX generated by ξ, and ρ_ξ the density $x\to|\xi(x)|^2_x\cdot\rho(x)$.

(e) Section IV is devoted to the study of $\Pi^{\xi}_{\rho\,dm}$ in a general case, namely when X is not assumed to be endowed with a Riemannian structure and for continuous densities.

As for the $U_{\rho,M}$, it is proved that $\Pi^{\xi}_{\rho\,dm}$ is irreducible when $\dim(X)\geqslant 3$; conditions on unitary equivalence of such representations are also given.

Some problems remain open:

In the case $\dim(X)=1$, $\Pi^{\xi}_{\rho\,dm}\sim U_{\rho_\xi,TX}$, so that $\Pi^{\xi}_{\rho\,dm}$ is reducible; what happens for $\dim(X)=2$?

Do the results about irreducibility of the $\Pi^{\xi}_{\rho\, dm}$ remain true when we take for ξ a nonvanishing vectorfield $dm(x)$-square measurable?

What are the results about irreducibility of $\Pi^{\xi}_{\rho\, dm}$ when, instead of a continuous density, ρ is only a locally $dm(x)$-measurable and strictly positive function on X, or when ξ is such that $\xi(x)=0$ for some x in X?

When X is not Riemannian, do nonlocal unitary representations exist (and, if possible, irreducible!) other than the $\Pi^{\xi}_{\rho\, dm}$?

I. Preliminary Technical Results about Dirichlet Forms and Related Gaussian Measures

(a) Let $B \neq \varnothing$ be an open ball or a bounded open cube in $\mathbb{R}^n$, and let A be a continuous mapping from B into the space of $n \times n$ real symmetric matrices, such that, for all x in B, $A(x) = (a_{i,j}(x))$ is positive definite. As all norms in $\mathbb{R}^n$ are equivalent, from the continuity of A it follows that it exists an open cube $\Omega \neq \varnothing$ with closure $\bar{\Omega} \subset B$, and two real numbers $m_A(\Omega) > 0$ and $M_A(\Omega) > 0$ such that, for all x in Ω

$$m_A(\Omega)\, \mathbb{1} \leqslant A(x) \leqslant M_A(\Omega)\, \mathbb{1}, \tag{1}$$

where $\mathbb{1}$ is the $n \times n$ unit matrix. Moreover, if Ω' is an open cube contained in Ω, one has $m_A(\Omega') \geqslant m_A(\Omega)$ and $M_A(\Omega') \leqslant M_A(\Omega)$.

Let dx be the Lebesgue measure on Ω, let $W_2^1(\Omega)$ be the Sobolev space

$$W_{2,1}(\Omega) = \left\{ u \in L^2(\Omega, dx) \Big/ \int_\Omega \sum_{i=1}^{i=n} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \|u\|^2_{2,1} < +\infty \right\}, \tag{2}$$

where the derivative are taken in the distributional sense, and let $\dot{W}_2^1(\Omega)$ be the completion of the Schwartz space $\mathscr{D}(\Omega, \mathbb{R})$ of $\mathbb{R}$-valued and compactly supported C^∞-mappings on Ω, according to the Sobolev norm $\| \ \|_{2,1}$. The matrix-valued mapping A gives rise to an inner product on $\mathscr{D}(\Omega, \mathbb{R})$ (in fact, in $\dot{W}_{2,1}(\Omega)$) given by

$$(u, v) \to \tilde{A}(u, v) = \int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx, \tag{3}$$

and to the generalized Laplacian L_A, such that

$$\langle -L_A u, v \rangle = \tilde{A}(u, v), \qquad u \in \dot{W}_{2,1}(\Omega), v \in \mathscr{D}(\Omega, \mathbb{R}). \tag{4}$$

Moreover, A gives rise to the standard Gaussian measure μ_A on $\mathscr{D}'(\Omega)$ with Fourier transform

$$\hat{\mu}_A \colon u \to \hat{\mu}_A(u) = \exp(-\tfrac{1}{2}\tilde{A}(u, u)). \tag{5}$$

LEMMA 1. *Let A and Ω be as above.*

(i) *$\tilde{A}$ is a Dirichlet form on $L^2(\Omega, dx)$.*

(ii) *$-L_A$ is invertible.*

Proof. (i) From (1) and (3) it follows that for all u in $\mathring{W}_{2,1}(\Omega)$ $m_A(\Omega)\|u\|_{2,1}^2 \leqslant \tilde{A}(u,u) \leqslant M_A(\Omega)\|u\|_{2,1}^2$; as the components of A are bounded measurable functions on Ω, the assertion follows from [6].

(ii) From the same reasons it follows that the Dirichlet problem for the elliptic equation $\langle -L_A u, v\rangle = \langle f, v\rangle$ has nice data, and hence, from Theorems 5.1 and 5.2 of [8], it follows that $-L_A^{-11}$ is well defined on $\mathscr{D}(\Omega, \mathbb{R})$.

LEMMA 2. *Let B be an open ball in $\mathbb{R}^n$, let A be a continuous mapping on B and taking values in the space of real symmetric and positive definite $n \times n$ matrices, let ρ_1 and ρ_2 be two different continuous and strictly positive densities on B, and let $A_k = \rho_k A$, $k = 1, 2$. A bounded open cube $\Omega \neq \varnothing$ exists such that $\bar{\Omega} \subset B$ and having the following property: a Borel set P in $\mathscr{D}'(\Omega)$ exists such that $\mu_{A_1}(P) = 0$ and $\mu_{A_2}(P) = 1$.*

Proof. As $\rho_1 \neq \rho_2$, it exists x_0 in B such that $\rho_1(x_0) \neq \rho_2(x_0)$; without loss of generality, one can assume that $\varepsilon = \rho_2(x_0) - \rho_1(x_0)$ is >0, and that $A(x_0) = \mathbb{1}$. Now, due to the continuity of A, we can find an open neighbourhood Ω° of x_0 in B such that, for all x in Ω°

$$m_A(\Omega^\circ)\,\mathbb{1} \leqslant A(x) \leqslant M_A(\Omega^\circ)\,\mathbb{1},$$

and such that

$$M_A(\Omega^\circ)/m_A(\Omega^\circ) < \rho_2(x_0)/\rho_1(x_0);$$

particularly one has

$$\rho_1(x_0)[M_A(\Omega^\circ) - m_A(\Omega^\circ)] < \varepsilon m_A(\Omega^\circ). \tag{6}$$

From (6) it follows that one can find a sufficiently large integer p such that

$$\begin{gathered}(p-1)\,m_A(\Omega^\circ) > M_A(\Omega^\circ),\\ \rho_1(x_0)[M_A(\Omega^\circ) - m_A(\Omega^\circ)] < \varepsilon[m_A(\Omega^\circ) - (m_A(\Omega^\circ) + M_A(\Omega^\circ))/p],\end{gathered} \tag{7}$$

and hence such that

$$[\rho_1(x_0) + \varepsilon/p]\,M_A(\Omega^\circ) < [\rho_1(x_0) + (p-1)/p \cdot \varepsilon]\,m_A(\Omega^\circ). \tag{8}$$

Moreover, from the continuity of ρ_1 and ρ_2, it follows that one can find an open cube Ω such that

$$\{x_0\} \subset \Omega \subset \bar{\Omega} \subset \Omega^\circ \subset B,$$

and such that for all x in Ω

$$|\rho_1(x)-\rho_1(x_0)| < \varepsilon/p \qquad \text{and} \qquad |\rho_2(x)-\rho_2(x_0)| < \varepsilon/p;$$

as $\varepsilon = \rho_2(x_0) - \rho_1(x_0)$ it follows that, for all x in Ω

$$\rho_1(x) < \rho_1(x_0) + \varepsilon/p < \rho_1(x_0) + \varepsilon(p-1)/p = \rho_2(x_0) - \varepsilon/p < \rho_2(x).$$

Hence

$$\sup_{x\in\Omega} \rho_1(x) \leqslant \rho_1(x_0) + \varepsilon/p < \rho_1(x_0) + \varepsilon(p-1)/p \leqslant \inf_{x\in\Omega} \rho_2(x). \tag{9}$$

Now, as $\Omega \subset \Omega^\circ$, one has $m_A(\Omega^\circ) \leqslant m_A(\Omega) \leqslant M_A(\Omega) \leqslant M_A(\Omega^\circ)$; from the inequality $m_A(\Omega)\,\mathbb{1} \leqslant A(x) \leqslant M_A(\Omega)\,\mathbb{1}$, $x \in \Omega$, it follows from (9) and from the fact that $A_k = \rho_k A$, $k = 1, 2$, that, for all x in Ω

$$A_1(x) \leqslant \theta_1 \mathbb{1} < \theta_2 \mathbb{1} < A_2(x), \tag{10}$$

where

$$\theta_1 = [\rho_1(x_0) + \varepsilon/p]\, M_A(\Omega) < \theta_2 = [\rho_1(x_0) + \varepsilon(p-1)/p]\, m_A(\Omega).$$

By Proposition A.1 of [3], inequality (10) is a sufficient condition to assert the existence of a Borel set P in $\mathscr{D}'(\Omega)$ such that $\mu_{A_1}(P) = 0$ and $\mu_{A_2}(P) = 1$.

LEMMA 3. *Let B be an open ball in $\mathbb{R}^n$, $n \geqslant 3$, and let A, ρ_1, ρ_2, and Ω be as in Lemma 2. A Borel set Q in $\mathscr{D}'(\Omega)$ exists such that*

(i) $\mu_{A_1}(Q) = 1$;

(ii) $\mu_{A_k}(Q + \alpha\delta_x) = 0$, $k = 1, 2$, *for all x in Ω and all real $\alpha \neq 0$, where $A_k = \rho_k A$, $k = 1, 2$, and where δ_x is the Dirac measure at x.*

Proof. Lemma 3 is a strengthening of Theorem 4.1 of [2], where the same result is proved under the stronger assumption of the smoothness of A, ρ_1, ρ_2 (instead of the assumption of continuity). In the proof of that Theorem 4.1, the condition C^∞ appears in only two points.

(a) The first point, where the hypothesis of smoothness is used, is that which claims the existence of the inverse of the operator $-L_{A_k}$ and of its kernel, $k = 1, 2$; but, as was shown in the Lemma 1, the assumption of continuity of A, ρ_1, and ρ_2 is sufficient to claim that existence.

(b) The second and last point is that which uses the disjointness of μ_{A_1} and μ_{A_2} (which was proved in the proposition of Section 4 of [10] in the case of A, ρ_1, ρ_2 of C^∞-class). But Lemma 2 asserts that the disjointness of μ_{A_1} and μ_{A_2} remains true with only the assumption of continuity of A, ρ_1, and ρ_2. The proof of Lemma 3 is hence achieved.

II. The Generalized Energy Representations $U_{\rho,M}$ of a Current Group

(a) Throughout this paper, G is a compact semisimple Lie group with Lie algebra $\mathscr{G}$ endowed with the scalar product $\langle\ ,\ \rangle_G$ and norm $|\ |_G$ given by $-K$, where K is the Killing form of $\mathscr{G}$, X is a C^∞-manifold, and $\mathscr{D}(X, G)$ denotes the set of compactly supported C^∞-mappings on X and taking values in G, endowed with the structure of topological group given by the Schwartz topology of $\mathscr{D}(X, \mathscr{G}) = C_0^\infty(X, \mathscr{G})$ and the pointwise multiplication.

(b) In this section the manifold X is supposed to be endowed with a Riemannian structure; m denotes the induced volume measure, and for each x in X, $(\ ,\ )_x$ and $|\ |_x$ denote the induced inner product and the induced norm on the tangent space T_xX.

Let $\Lambda(X)$ be the set of strictly positive and locally $dm(x)$-measurable functions on X; each ρ in $\Lambda(X)$ gives rise to an inner product $(\ ,\ )_\rho$ on the space $\mathscr{D}_1(X, \mathscr{G})$ of compactly supported smooth 1-forms on X and taking values in $\mathscr{G}$ by

$$(\omega, \omega')_\rho = \int_X \operatorname{tr}[\omega^*(x) \cdot \omega'(x)]\, \rho(x)\, dm(x), \tag{11}$$

where, for each x in X, $\omega^*(x)\colon \mathscr{G} \to T_xX$ is the adjoint of the element $\omega(x)$ of $\operatorname{Hom}(T_xX, \mathscr{G})$.

(c) Let $M = \bigcup_{x\in X} M_x$ be a nonzero subbundle of the tangent bundle $TX = \bigcup_{x\in X} T_xX$ of X.

To each such subbundle M the subspace

$$\mathscr{D}_1(X, \mathscr{G})_M = \{\omega \in \mathscr{D}_1(X, \mathscr{G}) / \forall x \in X, \omega(x)|_{M_x^\perp} = 0\}, \tag{12}$$

is associated, where $M_x^\perp$ is the orthogonal complement of M_x in T_xX. Here $\omega \to \omega_M$, where ω_M is given by

$$\omega_M(x)|_{M_x} = \omega(x)|_{M_x} \qquad \text{and} \qquad \omega_M(x)|_{M_x^\perp} = 0, \qquad x \in X, \tag{13}$$

is the orthogonal projection of $\mathscr{D}_1(X, \mathscr{G})$ on $\mathscr{D}_1(X, \mathscr{G})_M$, according to the inner product $(\ ,\ )_\rho$, for all ρ in $\Lambda(X)$.

(d) Let X, G, M and ρ be as above. For all g in $\mathscr{D}(X, G)$ the operator $V_M(g)$ on $\mathscr{D}_1(X, \mathscr{G})_M$, such that for any 1-form ω, $V_M(g)\omega$ is the 1-form

$$x \to V_M(g)\, \omega(x) = \operatorname{Ad} g(x) \circ \omega(x), \tag{14}$$

where Ad is the adjoint representation of G in $\mathscr{G}$, leaves invariant the inner product $(\ ,\ )_\rho$. We extend V_M by transposition into a representation again denoted V_M, of $\mathscr{D}(X, G)$ in the dual space $\mathscr{D}_1'(X, \mathscr{G})_M$ by

$$\langle V_M(g)\chi, \omega\rangle = \langle \chi, V_M(g^{-1})\omega\rangle. \tag{15}$$

Moreover the Maurer–Cartan cocycle $b: g \to dg \cdot g^{-1}$, which is a 1-cocycle for the representation $V = V_{TX}$ [9] gives rise to a 1-cocycle b_M for V_M by

$$b_M: g \to b(g)_M. \tag{16}$$

We have hence all the data to get (following the general procedure described in [5, 9]), a unitary representation of exponential type $U_{\rho,M}$ of $\mathscr{D}(X, G)$. More precisely let $\mu_{\rho,M}$ be the standard Gaussian measure on $\mathscr{D}'_1(X, \mathscr{G})_M$ according to the scalar product $(\ ,\ )_\rho$, i.e., with Fourier transform $\hat{\mu}_{\rho,M}$ given by

$$\hat{\mu}_{\rho,M}(\omega) = \exp[-\tfrac{1}{2}(\omega, \omega)_\rho]; \tag{17}$$

$U_{\rho,M}$ is a unitary representation of order 1 of $\mathscr{D}(X, G)$ in $L^2(\mathscr{D}'_1(X, \mathscr{G})_M; d\mu_{\rho,M})$ such that for g in $\mathscr{D}(X, G)$, Φ in $L^2(\mathscr{D}'_1(X, \mathscr{G})_M; d\mu_{\rho,M})$, χ in $\mathscr{D}'_1(X, \mathscr{G})_M$

$$U_{\rho,M}(g)\, \Phi(\chi) = \exp\{i\langle b_M(g), \chi\rangle\}\, \Phi[V_M(g^{-1})\chi]. \tag{18}$$

Such representations have been constructed and studied only in the case of C^∞-densities, i.e., in the case ρ in $\Lambda^\infty(X) = \Lambda(X) \cap C^\infty(X)$, at first in [7] for $G = SU(2)$ and $M = TX$, then in [10, 11] for any G and M. In [1], the representations $U_\rho = U_{\rho,TX}$, $\rho \in \Lambda^\infty(X)$ were called energy representations.

The aim of this section is to prove that the results about the irreducibility and about the inequivalency of the $U_{\rho,M}$ given in [2, 10] remain true with only continuous densities.

THEOREM 1. *Let X be a Riemannian manifold with* $\dim(X) \geqslant 3$, *endowed with a Riemannian structure, let G be a compact semisimple Lie group, and let $\Lambda^0(X)$ be the set of continuous and strictly positive densities on X.*

(i) *For any ρ in $\Lambda^0(X)$ and any nonzero subbundle M of TX the representation $U_{\rho,M}$ is irreducible.*

(ii) *Let ρ_1, ρ_2 be in $\Lambda^0(X)$ and let M_1, M_2 be two nonzero subbundles of TX; U_{ρ_1,M_1} is equivalent to U_{ρ_2,M_2} if and only if $M_1 = M_2$ and $\rho_1 = \rho_2$.*

Proof. (a) Let us consider first the energy representation $U_\rho = U_{\rho,TX}$, ρ in $\Lambda^0(X)$. The irreducibility of U_ρ is proved in Theorem 2.1 of [2] in the case of a C^∞-density ρ. In Theorem 2.3 of [2] it is also proved that for C^∞-densities ρ_1, ρ_2, U_{ρ_1}, and U_{ρ_2} are equivalent if and only if $\rho_1 = \rho_2$. In [2] the proofs of these two results are given in two parts.

The first part, called "algebraic part of the proofs" does not require the smoothness of the densities.

The second part requires the soothness of the densities in one point: in the proof of Theorem 4.1 of the same paper; but, as it has been shown in Lemma 3, Section 1, results of Theorem 4.1 remain valid with only continuous densities. It follows that assertions (i) and (ii) are true in the case $M = M_1 = M_2 = TX$, and densities ρ, ρ_1, ρ_2 in $\Lambda^0(X)$.

(b) Let M be a nonzero subbundle of TX, let ρ be in $\Lambda^0(X)$, and let $M^\perp$ be the orthogonal subbundle of M in TX. It is easy to see that, for all g in $\mathscr{D}(X, G)$,

$$U_{\rho,TX}(g) = U_{\rho,M}(g) \otimes U_{\rho,M^\perp}(g).$$

The irreducibility of $U_{\rho,M}$ follows hence from the irreducibility of $U_\rho = U_{\rho,TX}$, so that (i) is proved.

(c) Assertion (ii) has been given in [10] in the case of C^∞-densities.

Let $\mathscr{Z}$ be a Cartan subalgebra of $\mathscr{G}$ and let Φ be the set of elements of $\mathscr{D}'(X, \mathscr{Z})$ which are of the form

$$\chi\colon U \to \chi(U) = \sum_{i=1}^{i=p} \alpha_i(U(x_i)),$$

for some x_i in X and for some α_i in the set Δ of roots of the pair $(\mathscr{G}, \mathscr{Z})$.

By identification of Φ with the disjoint union of the sets $(X \times \Delta)^p_{\mathrm{sym}}$, $p = 0, 1, 2, \ldots$, we endow Φ with a natural Borel structure. Each density ρ_k, $k = 1, 2$, gives rise to a Poisson measure ν_k on Φ in the following way: for each integer $p = 0, 1, 2, \ldots$, the restriction ν_k^p of ν_k to the set $(X \times \Delta)^p_{\mathrm{sym}}$ is equal to

$$[\rho_k(x)\, dm(x) \otimes N]^{\otimes p},$$

where N is the counting measure [5, Chap. I and Appendix E]. Let m_k be the standard Gaussian measure on $\mathscr{D}'(X, \mathscr{Z})$ associated to the inner product $\langle\ ,\ \rangle^{(k)}$ on $\mathscr{D}(X, \mathscr{Z})$ given by $(U, V) \to \langle U, V\rangle^{(k)} = (dU_{M_k}, dV_{M_k})_{\rho_k}$ (that last inner product being given in Eqs. (11) and (14)).

The Fourier transform $\hat{m}_k$ of m_k and the Fourier transform $\hat{\mu}_{\rho_k, M_k}$ of the standard Gaussian measure μ_{ρ_k, M_k} on $\mathscr{D}'_1(X, \mathscr{G})_{M_k}$ given in (17) are related by

$$\hat{m}_k(U) = \hat{\mu}_{\rho_k, M_k}(dU), \qquad U \in \mathscr{D}(X, \mathscr{Z}).$$

Now, let W_k be the representation of the abelian group $\mathscr{D}(X, \mathscr{Z})$ in $L^2[\mathscr{D}'_1(X, \mathscr{G})_{M_k}, d\mu_{\rho_k, M_k}]$ given by

$$W_k(U) = U_{\rho_k, M_k}(\exp U), \qquad k = 1, 2.$$

It follows from Proposition 3.1 of [2] that the spectral measure of W_k is the convolution $m_k * \nu_k$, and hence, as in the case of C^∞-densities (cf. [2]), it follows that $U_{\rho_1, M_1} \sim U_{\rho_2, M_2}$ if and only if $M_1 = M_2$ and if $m_1 * \nu_1$ is equivalent to $m_2 * \nu_2$. As ν_1 and ν_2 are equivalent the proof is hence reduced to solve the following problem:

Let ν be a Poisson measure such that $\nu(0) = 0$, and let M be a subbundle of TX; under what conditions are the convolutions $m_1 * \nu$ and $m_2 * \nu$ disjoint? In the case of C^∞-densities, the answer is given by Lemma 3.7 of [2]: they are disjoint as soon as $\rho_1 \neq \rho_2$. As that lemma follows directly from Theorem 4.1 of [2], and as the validity of that theorem has been extended to the case of continuous densities in our Lemma 3, Section I, the conclusion of Lemma 3.7 of [2] remains true with only continuous densities. The proof is hence achieved.

Note. Theorems 1 and 2 show that the family of (class of) generalized energy representations $U_{\rho, M}$ with continuous densities is strictly larger than the family of energy representations constructed by Ismagilov in [7] and by Vershik *et al.* in [10] (i.e., with only C^∞-densities). Moreover, it is not too difficult to see that if it could be proved that Lemma 2 of Section I remains true for bounded and locally $dm(x)$-measurable strictly positive functions on X, then Theorems 1 and 2 should remain true for such densities.

III. Construction of Partial Energy Representations

In this section we shall assume that the manifold X belongs to the class $\mathfrak{X}_0$ of smooth manifolds for which the Euler number is zero. The aim of this part is to prove that, for such a manifold X endowed with a volume measure dm, it is possible to get many nonlocal unitary representations of order 1 of $\mathscr{D}(X, G)$, without needing to select a Riemannian structure on X.

(a) Let X be a manifold in the class $\mathfrak{X}_0$; a well-known result [9, Corollary 39.8] asserts that this condition is equivalent to the fact that the set $\mathscr{L}^0(X)$ of nonvanishing continuous vectorfields on X is not empty. Hence, as soon as $\dim(X) \geqslant 2$, each element ξ of $\mathscr{L}^0(X)$ gives rise to a nontrivial subbundle $[\xi] = \bigcup_{x \in X} R\xi(x)$ of the tangent bundle TX, with 1-dimensional fibers.

We consider now the space $C_0(X, \mathscr{G})$ of compactly supported and continuous functions on X and taking values in the Lie algebra $\mathscr{G}$ of a compact semisimple Lie group G. For any ρ in $\Lambda(X)$, we define an inner product $\langle\ ,\ \rangle_\rho$ on $C_0(X, \mathscr{G})$ by

$$\langle U, V \rangle_\rho = \int_X \langle U(x), V(x) \rangle_G \cdot \rho(x)\, dm(x), \tag{19}$$

where, as in Section II, $\langle\ ,\ \rangle_G$ is the scalar product on $\mathscr{G}$ invariant by the adjoint representation Ad of G in $\mathscr{G}$. Let ξ be in $\mathscr{L}^0(X)$; ξ gives rise to a continuous linear mapping $\tilde{\xi}$ from $\mathscr{D}_1(X, \mathscr{G})$ into $C_0(X, \mathscr{G})$ such that for all ω in $\mathscr{D}_1(X, \mathscr{G})$

$$\tilde{\xi}(\omega)(x) = \omega(x)(\xi(x)), \qquad x \in X. \tag{20}$$

For all g in $\mathscr{D}(X, G)$ the operator $A(g)$ on $C_0(X, \mathscr{G})$ given by

$$(A(g)U)(x) = \operatorname{Ad} g(x)(U(x)), U \in C_0(X, \mathscr{G}), \qquad x \in X, \tag{21}$$

leaves invariant the inner products $\langle\ ,\ \rangle_\rho$, for all ρ in $\Lambda(X)$; hence one has a unitary representation $A: g \to A(g)$ of $\mathscr{D}(X, G)$ in the prehilbertian space $[C_0(X, \mathscr{G}), \langle\ ,\ \rangle_\rho]$. Moreover,

LEMMA 4. *Let ξ be in $\mathscr{L}^0(X)$; for all g in $\mathscr{D}(X, G)$ one has $\tilde{\xi} \circ V(g) = A(g) \circ \tilde{\xi}$.*

Proof. Let us recall that the representation V of $\mathscr{D}(X, G)$ in $\mathscr{D}_1(X, \mathscr{G})$ is given by

$$(V(g)\omega)(x) = \operatorname{Ad} g(x) \circ \omega(x), \qquad \omega \in \mathscr{D}_1(X, \mathscr{G}), x \in X.$$

Hence

$$\begin{aligned}[(\tilde{\xi} \circ V(g))(\omega)](x) &= (V(g)\omega)(x)(\xi(x)) = [\operatorname{Ad} g(x) \circ \omega(x)](\xi(x)) \\ &= \operatorname{Ad} g(x)[\omega(x)(\xi(x))] = [A(g)(\tilde{\xi}(\omega))](x).\end{aligned}$$

Let $\operatorname{Im}(\tilde{\xi})$ be the subspace $\tilde{\xi}(\mathscr{D}_1(X, \mathscr{G}))$ of $C_0(X, \mathscr{G})$, let A_ξ be the mapping assigning to each g in $\mathscr{D}(X, G)$ the restriction of the operator $A(g)$ to $\operatorname{Im}(\tilde{\xi})$, and let $b^\xi = \tilde{\xi} \circ b$, where b is the Maurer–Cartan cocycle (Section II, (16)). It follows from Lemma 4 that:

COROLLARY. *Let ρ be in $\Lambda(X)$ and ξ be in $\mathscr{L}^0(X)$; A_ξ is a unitary representation of $\mathscr{D}(X, G)$ in the prehilbertian space $[\operatorname{Im}(\tilde{\xi}), \langle\ ,\ \rangle_\rho]$ for which b^ξ is a 1-cocycle.*

(b) Now, following the general procedure described in [5] to get unitary representations of exponential type, let us extend by transposition the representation A_ξ to the algebraic dual space $\operatorname{Im}(\tilde{\xi})^*$, and ρ being chosen in $\Lambda(X)$, let μ_ρ be the Gaussian measure on $\operatorname{Im}(\tilde{\xi})^*$ with Fourier transform $\hat{\mu}_\rho: U \to \exp[-\frac{1}{2}\langle U, U\rangle_\rho]$; then we get a unitary representation $\Pi^\xi_{\rho\, dm}$, nonlocal and of order 1, of $\mathscr{D}(X, G)$ in $L^2[\operatorname{Im}(\tilde{\xi})^*; d\mu_\rho]$ such that, for all g in $\mathscr{D}(X, G)$, all Φ in $L^2[\operatorname{Im}(\tilde{\xi})^*; d\mu_\rho]$, all χ in $\operatorname{Im}(\tilde{\xi})^*$

$$\Pi^\xi_{\rho\, dm}(g)\, \Phi(\chi) = \exp\{i\langle b^\xi(g), \chi\rangle\}\, \Phi(A_\xi(g^{-1})\chi). \tag{22}$$

(c) We have achieved then nonlocal and order 1 unitary representations of $\mathscr{D}(X, G)$ which do not require a Riemannian structure on X, contrary to the ones constructed before.

A first problem is the following: What is the relationship between the generalized energy representation $U_{\rho,M}$ and the "partial energy representations" $\Pi^{\xi}_{\rho\, dm}$, when dm is the volume measure given by a Riemannian structure on X? The answer is given by

THEOREM 2. *Let X be in the class $\mathfrak{X}_0$ endowed with a Riemannian structure and let $dm(x)$ be the corresponding volume measure. An element ξ of $\mathscr{L}^0(X)$ and an element ρ of $\Lambda(X)$ being given, the representation $\Pi^{\xi}_{\rho\, dm}$ is unitarily equivalent to the generalized energy representation $U_{\rho_\xi, [\xi]}$, where ρ_ξ is the density $x \to \rho_\xi(x) = |\xi(x)|^2_x \cdot \rho(x)$.*

Proof. Let us recall that $[\xi]$ denotes the subbundle $\bigcup_{x\in X} \mathbb{R}\xi(x)$ of TX, and that $|\ |_x$ denotes, for all x in X, the Euclidean norm on T_xX induced by the selected Riemannian structure on X.

(a) Let us first prove that $\tilde{\xi}$ restricts oneself in an isometric bijection from the prehilbertian space $[\mathscr{D}_1(X, \mathscr{G})_{[\xi]}, (\ ,\ )_{\rho_\xi}]$ onto the prehilbertian space $[\mathrm{Im}(\tilde{\xi}), \langle\ ,\ \rangle_\rho]$. Let U be in $\mathrm{Im}(\tilde{\xi})$, and ω in $\mathscr{D}_1(X, \mathscr{G})$ such that $\tilde{\xi}(\omega) = U$. The orthogonal projection $\omega_{[\xi]}$ (cf. (13)) of ω onto the subspace $\mathscr{D}_1(X, \mathscr{G})_{[\xi]}$ is such that $\tilde{\xi}(\omega_{[\xi]}) = \tilde{\xi}(\omega) = U$; moreover each ω in $\mathscr{D}_1(X, \mathscr{G})_{[\xi]}$ is fixed, for all x in X, by its value at $\xi(x)$, hence by $\tilde{\xi}(\omega)$. It follows that $\tilde{\xi}$ restricts oneself in a bijection from $\mathscr{D}_1(X, \mathscr{G})_{[\xi]}$ onto $\mathrm{Im}(\tilde{\xi})$.

Now let ρ_ξ be the density: $x \to |\xi(x)|^2_x \cdot \rho(x)$. For ω in $\mathscr{D}_1(X, \mathscr{G})_{[\xi]}$ (completely fixed by the values $\omega(x)(\xi(x))$) for all x in X one has

$$\mathrm{tr}(\omega^*(x) \cdot \omega(x)) = \left\langle \omega^*(x) \cdot \omega(x) \left(\frac{1}{|\xi(x)|_x} \xi(x)\right); \frac{1}{|\xi(x)|_x} \xi(x) \right\rangle_G$$

$$= \frac{1}{|\xi(x)|^2_x} \langle \omega(x)(\xi(x)), \omega(x)(\xi(x)) \rangle_G.$$

Hence

$$\|\omega\|^2_{\rho_\xi} = \int_X \mathrm{tr}(\omega^*(x) \cdot \omega(x))\, \rho_\xi(x)\, dm(x)$$

$$= \int_X \langle \tilde{\xi}(\omega)(x), \tilde{\xi}(\omega)(x) \rangle_G\, \rho(x)\, dm(x)$$

$$= |\tilde{\xi}(\omega)|^2_\rho.$$

It follows that $\tilde{\xi}$ is an isometry from the prehilbertian space $[\mathscr{D}_1(X, \mathscr{G})_{[\xi]}, (\ ,\ )_{\rho_\xi}]$ onto the prehilbertian space $[\mathrm{Im}(\tilde{\xi}), \langle\ ,\ \rangle_\rho]$.

(b) Let us again denote $\tilde{\xi}$ the extension by transposition of $\tilde{\xi}$ which is a morphism from $\mathrm{Im}(\tilde{\xi})^*$ into $\mathscr{D}'_1(X, \mathscr{S})_{[\xi]}$ given by

$$\langle\tilde{\xi}(\chi), \omega\rangle = \langle\chi, \tilde{\xi}(\omega)\rangle, \qquad \chi \in \mathrm{Im}(\tilde{\xi})^*, \omega \in \mathscr{D}_1(X, \mathscr{S})_{[\xi]}.$$

It follows from Lemma 4, that for all g in $\mathscr{D}(X, G)$

$$\tilde{\xi} \cdot A_\xi(g) = V_{[\xi]}(g) \cdot \tilde{\xi}.$$

Now, let T_ξ be the operator from $L^2[\mathscr{D}'_1(X, \mathscr{S})_{[\xi]}; d\mu_{\rho_\xi, [\xi]}]$ into $L^2[\mathrm{Im}(\tilde{\xi})^*; d\mu_{\rho\, dm}]$, given by

$$(T_\xi \Phi)(\chi) = \Phi(\tilde{\xi}(\chi)).$$

From the above discussion it follows that T_ξ is an isometry of Hilbert spaces. Moreover, for all g in $\mathscr{D}(X, G)$, Φ in $L^2[\mathscr{D}'_1(X, \mathscr{S})_{[\xi]}: d\mu_{\rho_\xi, [\xi]}]$, and χ in $\mathrm{Im}(\tilde{\xi})^*$

$$T_\xi \cdot U_{\rho_\xi, [\xi]}(g)\, \Phi(\chi) = \exp\{i\langle\tilde{\xi}(\chi), b_{[\xi]}(g)\rangle\} \cdot \Phi[\tilde{\xi} \cdot V_{[\xi]}(g^{-1})\chi];$$

as $\langle\tilde{\xi}(\chi), b_{[\xi]}(g)\rangle = \langle\chi, \tilde{\xi}(b_{[\xi]}(g))\rangle = \langle\chi, \tilde{\xi}(b(g))\rangle = \langle\chi, b^\xi(g)\rangle$, and as $\tilde{\xi} \cdot V_{[\xi]}(g^{-1}) = A_\xi(g^{-1}) \cdot \tilde{\xi}$, it follows that

$$T_\xi \cdot U_{\rho_\xi, [\xi]}(g)\, \Phi(\chi) = \Pi^\xi_{\rho\, dm}(g)(T_\xi \Phi)(\chi).$$

Hence T_ξ intertwines $U_{\rho_\xi, [\xi]}$ and $\Pi^\xi_{\rho\, dm}$.

As a corollary one has:

COROLLARY. *Let X be in the class $\mathfrak{X}_0$, endowed with a Riemannian structure with volume measure $dm(x)$, and such that* $\dim(X) \geqslant 3$. *Let ρ, ρ_1, ρ_2 be in $\Lambda^0(X)$ and let ξ, ξ_1, ξ_2 be in $\mathscr{L}^0(X)$:*

(i) *$\Pi^\xi_{\rho\, dm}$ is an irreducible unitary representation of $\mathscr{D}(X, G)$;*

(ii) *$\Pi^{\xi_1}_{\rho_1 dm}$ and $\Pi^{\xi_2}_{\rho_2 dm}$ are unitary equivalent if and only if, for all x in X,*

$$\xi_2(x) = \pm \sqrt{\rho_1(x)/\rho_2(x)}\; \xi_1(x).$$

Proof. Condition (i) follows directly from Theorem 1(i) and from Theorem 2.

(ii) From Theorems 2 and 1(ii) one has $\Pi^{\xi_1}_{\rho_1 dm} \sim \Pi^{\xi_2}_{\rho_2 dm}$ if and only if $[\xi_1] = [\xi_2]$ and $\rho_{\xi_1} = \rho_{\xi_2}$. But $[\xi_1] = [\xi_2]$ if and only if a continuous mapping $\theta: X \to R$ exists such that $\xi_2 = \theta \cdot \xi_1$. From the equality $\rho_{\xi_1} = \rho_{\xi_2}$ it follows that for all x in X

$$|\xi_1(x)|^2_x \cdot \rho_1(x) = |\xi_2(x)|^2_x \cdot \rho_2(x) = |\theta(x)| \cdot |\xi_1(x)|\, \rho_1(x).$$

Hence the assertion is proved.

IV. STUDY OF $\Pi^{\xi}_{\rho\, dm}$ IN THE GENERAL CASE FOR CONTINUOUS DENSITIES

As in Section III, the manifold X is taken in the class $\mathfrak{X}_0$. The natural question arising now is the following: When X is not endowed with a Riemannian structure (in that case $U_{\rho,M}$ does not exist), does the corollary of Theorem 3 remain true?

The main goal of this section is to give an affirmative answer in the case of continuous densities.

(a) Let T be a maximal torus in G, with Lie algebra $\mathscr{Z}$, and let $\mathscr{Z}^{\perp}$ be the orthogonal complement of $\mathscr{Z}$ in $\mathscr{G}$. For a given ξ in $\mathscr{S}^0(X)$, $\mathrm{Im}(\tilde{\xi}, \mathscr{Z})$ and $\mathrm{Im}(\tilde{\xi}, \mathscr{Z}^{\perp})$ denote the subspaces $\tilde{\xi}(\mathscr{D}_1(X, \mathscr{Z}))$ and $\tilde{\xi}(\mathscr{D}_1(X, \mathscr{Z}^{\perp}))$ in $C_0(X, \mathscr{G})$. The related Gaussian measures on the algebraic duals $\mathrm{Im}(\tilde{\xi}, \mathscr{Z})^*$ and $\mathrm{Im}(\tilde{\xi}, \mathscr{Z}^{\perp})^*$, associated to the density ρ, element of $\Lambda^0(X)$, are respectively denoted $\mu_\rho^{\mathscr{Z}}$ and $\mu_\rho^{\mathscr{Z}\perp}$. It is easy to see that

$$L^2[\mathrm{Im}(\tilde{\xi})^*; d\mu_\rho] = L^2[\mathrm{Im}(\tilde{\xi}, \mathscr{Z})^*; d\mu_\rho^{\mathscr{Z}}] \otimes L^2[\mathrm{Im}(\tilde{\xi}, \mathscr{Z}^{\perp})^*; d\mu_\rho^{\mathscr{Z}\perp}], \quad (23)$$

and that, for all U in $\mathscr{D}(X, \mathscr{Z})$

$$b^{\xi}(\exp U) = \tilde{\xi}(dU). \quad (24)$$

It follows that the operators $\Pi^{\xi}_{\rho\, dm}(\exp U)$ have the form

$$\Pi^{\xi}_{\rho\, dm}(\exp U)\, \Phi(\chi) = \exp\{i\langle \tilde{\xi}(dU), \chi\rangle\}\, \Phi[A_{\xi}(-\exp U)\chi].$$

Hence, for all U in $\mathscr{D}(X, \mathscr{Z})$, one has

$$\Pi^{\xi}_{\rho\, dm}(\exp U) = W_{\mathscr{Z}}(\tilde{\xi}(dU)) \otimes W_{\mathscr{Z}^{\perp}}(\exp U), \quad (25)$$

where $W_{\mathscr{Z}}(\tilde{\xi}(dU))\, \Phi(\chi) = \exp\{i\langle \tilde{\xi}(dU), \chi\rangle\}\, \Phi(\chi)$, $\Phi \in L^2(\mathrm{Im}(\tilde{\xi}, \mathscr{Z})^*; d\mu_\rho^{\mathscr{Z}})$, $\chi \in \mathrm{Im}(\tilde{\xi}, \mathscr{Z})^*$, and where $W_{\mathscr{Z}^{\perp}}(\exp U)\, \Phi(\chi) = \Phi(A_{\xi}(-\exp U)\chi)$, with Φ in $L^2(\mathrm{Im}(\tilde{\xi}, \mathscr{Z}^{\perp})^*; d\mu_\rho^{\mathscr{Z}\perp})$, χ in $\mathrm{Im}(\tilde{\xi}, \mathscr{Z}^{\perp})^*$.

Now using the same argument as in Lemma 4, Section 4 of [9] and Section 3 of [2], one easily shows that one has the direct integral decomposition

$$\Pi^{\xi}_{\rho\, dm}(\exp U) = \int_{\Phi}^{\oplus} W_{\mathscr{Z}}(\tilde{\xi}(dU))\, e^{i\chi(U)}\, d\nu(\chi), \quad (26)$$

where Φ, as in the proof of Theorem 1, Section II, is the set of elements of $\mathscr{D}'(X, \mathscr{Z})$ of the form

$$\chi\colon U \to \chi(U) = \sum_{i=1}^{i=p} \alpha_i(U(x_i)), \qquad p = 0, 1, 2, \ldots,$$

α_i in the set Δ of roots of the pair $(\mathscr{G}, \mathscr{Z})$, x_i in X, and where ν is the Poisson measure, such that by identification of Φ with $\bigcup_{p=0}^{\infty}(X \times \Delta)^p_{\mathrm{sym}}$, the restriction ν^p of ν on $(X \times \Delta)^p_{\mathrm{sym}}$ is given by

$$\nu^p = [\rho(x)\, dm(x) \otimes N]^{\otimes p}, \qquad N \text{ being the counting measure.}$$

By a standard argument, as for the study of the U_ρ in [2, Sect. 3, Proposition 3.1], one gets

LEMMA 5. *The spectral measure associated to the restriction of* $\Pi^{\xi}_{\rho\, dm}$ *to* $\mathscr{D}(X, T)$ *is the convolution* $\mu_\rho^{\mathscr{Z}} * \nu$.

We have now the crucial

LEMMA 6. *Let* ν_1 *and* ν_2 *be two mutually disjoint probability measures on* Φ; *then* $\mu_\rho^{\mathscr{Z}} * \nu_1$ *and* $\mu_\rho^{\mathscr{Z}} * \nu_2$ *are disjoint.*

Proof. This result is proved in Lemma 3.2 of [2] for a C^∞-density ρ, and where, instead of the Gaussian measure $\mu_\rho^{\mathscr{Z}}$, one takes the Gaussian measure m on $\mathscr{D}'(X, \mathscr{Z})$ associated to the inner product $(U, V) \to (dU, dV)_\rho$. For that proof, one shows first that one can assume that $\dim(\mathscr{Z}) = 1$. In [2] it is shown moreover that Lemma 3.2 is a direct consequence of Theorem 4.1 of the same paper about Gaussian measures given by the Dirichlet forms of the form

$$(u, v) \to \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx,$$

for some bounded open cube, the matrix valued function $A = (a_{i,j})$ being given by the Riemannian structure. Using an analogous way, we get such a Dirichlet form, where $a_{i,j} = a_i \cdot a_j$, the a_i being the components of ξ in Ω

$$\xi = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}.$$

In fact, let u and v be in $\mathscr{D}(\Omega, \mathbb{R})$

$$\begin{aligned}(u, v)_\rho &= \langle \tilde{\xi}(du), \tilde{\xi}(dv) \rangle \\ &= \int_\Omega du(x)(\xi(x)) \cdot dv(x)(\xi(x))\, dx \\ &= \int_\Omega \left(\sum_i a_i(x) \frac{\partial u}{\partial x_i}\right)\left(\sum_j a_j(x) \frac{\partial v}{\partial x_j}\right) dx \\ &= \int_\Omega \sum_{i,j} a_i(x)\, a_j(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx.\end{aligned}$$

Now, applying Lemma 3 of the Section I (which extends the validity of Theorem 4.1 of [2] to continuous densities) in the same manner as in the proof of Lemma 3.2 of [2], we get our Lemma 6.

THEOREM 3. *Let X be a manifold in the class $\mathfrak{X}_0$ endowed with a volume measure $dm(x)$, and let G be a compact semisimple Lie group. If* $\dim(X) \geqslant 3$, *for all ξ in $\mathscr{L}^0(X)$ and all ρ in $\Lambda^0(X)$, the representation $\Pi^{\xi}_{\rho\, dm}$ is irreducible.*

Proof. (a) By Lemma 6 and [11, Lemma 2, Sect. 5], it follows that the commutant of $\Pi^{\xi}_{\rho\, dm}(\mathscr{D}(X, T))$ is contained in the set of decomposable operators of the integral decomposition (26). Hence, using the proof of Theorem 2.1 of [2], one easily proves that the restriction of $\Pi^{\xi}_{\rho\, dm}$ to the cyclic component of the element $\Phi^0 \equiv 1$ of $L^2(\mathrm{Im}(\tilde{\xi})^*; d\mu_\rho)$ is irreducible.

(b) The proof of the irreducibility of $\Pi^{\xi}_{\rho\, dm}$ is hence reduced to the proof of the cyclicity of Φ^0. From Corollary 3.4 of [2] it suffices to prove that the set

$$\{A_\xi(g)[\tilde{\xi}(dU)],\ g \in \mathscr{D}(X, G),\ U \in \mathscr{D}(X, \mathscr{G})\}$$

is total in $\mathrm{Im}(\tilde{\xi})$. But, from Lemma 3 of Sect. 5 of [11], one knows that the set $\{V(g)(dU), g \in \mathscr{D}(X, G), U \in \mathscr{D}(X, G)\}$ is total in $\mathscr{D}_1(X, \mathscr{G})$. The assertion follows from the fact that $\tilde{\xi}(\mathscr{D}_1(X, \mathscr{G})) = \mathrm{Im}(\tilde{\xi})$.

THEOREM 4. *Let X be in $\mathfrak{X}_0$ with* $\dim(X) \geqslant 3$, *endowed with a volume measure m, and a compact semisimple Lie group G being given, let ξ_1, ξ_2 be in $\mathscr{L}^0(X)$ such that $[\xi_1] = [\xi_2]$, and let ρ_1, ρ_2 be in $\Lambda^0(X)$. $\Pi^{\xi_1}_{\rho_1\, dm}$ and $\Pi^{\xi_2}_{\rho_2\, dm}$ are unitarily equivalent if and only if for all x in X one has*

$$\xi_2(x) = \pm \sqrt{\rho_1(x)/\rho_2(x)} \cdot \xi_1(x).$$

Proof. (a) $[\xi_1] = [\xi_2]$ if and only if a continuous mapping $\lambda: X \to \mathbb{R}$ exists such that for all x in X

$$\xi_2(x) = \lambda(x)\, \xi_1(x).$$

Because of the continuity of λ and the fact that $\xi_2(x) \neq 0$ for all x in X, either λ or $-\lambda$ is strictly positive.

(b) Let us consider the Dirichlet forms induced by the pairs (ξ_1, ρ_1) and (ξ_2, ρ_2) on some bounded open cube Ω of $\mathbb{R}^n$, $n = \dim(X)$: if $a_1, a_2, \ldots, a_n$ are component functions of ξ_1 in Ω, these Dirichlet forms are α_1, α_2 such that

$$\alpha_1(u, v) = \int_\Omega \sum_{i,j} \rho_1(x)\, a_i(x)\, a_j(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx$$

and

$$a_2(u, v) = \int_\Omega \sum_{i,j} \rho_2(x)\, \lambda^2(x)\, a_i(x)\, a_j(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx,$$

which shows that the first is associated to the density ρ_1, and the second to the density $\rho_2 \cdot \lambda^2$. The proof is now analogous to that of Theorem 1(ii): as Lemma 3.7 of [2] is a corollary of Theorem 4.1 of [2], the extension of which is our Lemma 3 in Section I, Lemma 3.7 of [2] remains true for continuous densities. Hence it follows that if the densities ρ_1 and $\rho_2 \cdot \lambda^2$ are different, the Gaussian measures $\mu^{\mathfrak{T}}_{\rho_1}$ and $\mu^{\mathfrak{T}}_{\rho_2 \cdot \lambda^2}$, for any Cartan subalgebra of $\mathscr{G}$, have the following property: for any bounded measure ν on $\Phi = \bigcup_{p=0}^{\infty} (X \times \Delta)^p_{\text{sym}}$, $\mu^{\mathfrak{T}}_{\rho_1} * \nu$ is disjoint from $\mu^{\mathfrak{T}}_{\rho_2 \cdot \lambda^2} * \nu$ and from $\mu^{\mathfrak{T}}_{\rho_2 \cdot \lambda^2}$. One deduces that $\Pi^{\xi_1}_{\rho_1 dm}$ and $\Pi^{\xi_2}_{\rho_2 dm}$ are equivalent if and only if $\rho_1 = \rho_2 \cdot \lambda^2$, hence, taking into account part (a) of the proof, if and only if

$$\lambda(x) = \pm \sqrt{\rho_1(x)/\rho_2(x)} \qquad \text{for all } x \text{ in } X.$$

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